ON THE CIRCLES OF APOLLONIUS.

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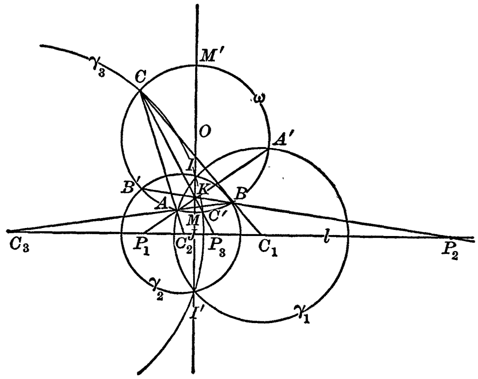
DEFINITION. The interior and exterior bisectors of the angles A, B, C of a triangle ABC meet the opposite sides in the points U, U'; V, V'; W, W' respectively. The circles described on the segments UU', VV', WW' as diameters, are called the *circles of Apollonius.*



Not all the lines considered are drawn in the figure.

(1) The bisectors CW, CW' being perpendicular to each other, the segment WW' subtends a right angle at the vertex C, therefore C is on the circle γ_3 described on WW' as diameter. Similarly for the points A and B with respect to the circles γ_1 and γ_2 described on UU' and VV' as diameters, and therefore:

The Apollonian circles pass through the respective vertices of the triangle.

(2) The bisectors CW, CW' separate harmonically the lines CA, CB,[1] hence any point S of the circle γ_3 is the center of the harmonic pencil of lines $S(ABWW')$, and since SW, SW' are perpendicular to each other, they are the bisectors of the angles at S[1]. According to a well-known theorem of plane geometry, we have for the triangles ABC and ABS,

$$\frac{CA}{CB} = \frac{AW}{BW} = \frac{AW'}{BW'}; \quad \frac{SA}{SB} = \frac{AW}{BW} = \frac{AW'}{BW'}.$$

Hence

$$\frac{SA}{SB} = \frac{CA}{CB}.$$

Now, if for any point S' in the plane we have $S'A/S'B = CA/CB$, the bisectors of the angles at S' will pass through W, W', according to the converse of the theorem cited, and the segment WW' will subtend at S' a right angle, *i. e.*, S' is on γ_3. These considerations may be repeated for the circles γ_1 and γ_2. Hence:

[1] John W. Russell, *Elementary Treatise on Pure Geometry*, pp. 18, 19.

The circle of Apollonius is the locus of a point the ratio of whose distances from two fixed points is constant.[1]

This property is frequently taken as the definition of the circle of Apollonius.

(3) From the definition and (1) it is evident that any two of the circles of Apollonius cross each other. Now, if I, I' are the points common to any two of these circles, say γ_2, γ_3, we have (2)

$$\frac{IA}{IC} = \frac{BA}{BC};\ \frac{IA}{IB} = \frac{CA}{CB}.$$

Hence

$$\frac{IB}{IC} = \frac{BA}{CA},\ i.\ e.,\ I \text{ is on } \gamma_1.$$

The same being true for the point I', we infer that

The three circles of Apollonius have two points in common.[2]

COROLLARY. *The centers of the three circles of Apollonius are collinear.*

(4) The points A, B, W, W' being harmonic (2) and WW' being a diameter of γ_3 (1), any circle passing through A, B cuts the circle γ_3 orthogonally.[3] Similarly for γ_1 and γ_2. Hence:

The circumcircle is orthogonal to each of the three circles of Apollonius.

(5) The radii OA, OB, OC of the circumcircle ω are the tangents drawn from the center O of ω to the respective circles of Apollonius (1, 4), and since $OA = OB = OC$, the point O belongs to the radical axis of the circles $\gamma_1, \gamma_2, \gamma_3$, which is the line joining the points I, I' (3) common to the three circles. Consequently:

The common chord of the three circles of Apollonius passes through the center of the circumcircle.[2]

COROLLARY.[3] *The points of intersection M, M' of the circumcircle with the chord $s \equiv II'$ are harmonically separated by the points I, I'.*

(6) The circles ω and γ_3 having the point C in common (1) and being orthogonal (4), the tangent to ω at C passes through the center C_3 of γ_3. The centers C_1, C_2 of the circles γ_1, γ_2 are determined in a like manner. Hence:

The tangents to the circumcircle at the vertices of the triangle meet the opposite sides of the triangle in the centers of the respective circles of Apollonius.[2]

(7) The circles ω and γ_3 intersect in C (1); let C' denote their other point of intersection. Since OC, OC' are the tangents drawn from O to γ_3 (4), the chord $s_3 \equiv CC'$ is the polar of O with respect to γ_3, and therefore meets $s \equiv$ chord II' at the harmonic conjugate K of O with respect to the pair of points I, I'. Similarly for the chords of intersection $s_1 \equiv AA'$, $s_2 \equiv BB'$ of ω with the circles γ_1 and γ_2. Therefore:

The three chords joining the three pairs of points of intersection of the circum-

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[1] Weber and Wellstein, *Encyklopädie der Elementar-Matematik*, Vol. 2, p. 250, sec. ed.

[2] John Casey, *Analytic Geometry*, p. 146, sec. ed. This MONTHLY, February, 1915, p. 59.

[3] Russell, *loc. cit.*, p. 26.

circle with each of the three circles of Apollonius, meet on the radical axis of the Apollonian circles.[1]

The common point K is called the *Lemoine* or *Symmedian point* of the triangle; the chords s_1, s_2, s_3 are called the *Symmedian lines* or *Symmedians* of the triangle;[2] $s \equiv II'$ is called the *Brocard diameter.*[3]

(8) The tangents from C to ω are the lines C_3C and C_3C' (4), C_3 is therefore the pole of $s_3 \equiv CC'$ with respect to ω; since C_3 is on AB, the line s_3 passes through the pole of AB with respect to ω, which pole is the point of intersection of the tangents to ω at A and B. Similarly for the chords s_1, s_2. Hence:

The Symmedian lines pass through the respective vertices of the triangle formed by the tangents to the circumcircle at the vertices of the given triangle.[4]

(9) The four lines s_1, s_2, s_3, s meeting at K (7) have their respective poles C_1, C_2, C_3, ∞ (8) with regard to the circumcircle ω on the polar of K with respect to ω, and the anharmonic ratio of the four lines is equal to the anharmonic ratio of the four points.[5] Hence:

A. The line of centers of the Apollonian circles is the polar of the Lemoine point with respect to the circumcircle.

The line is called the *Lemoine axis* or *line.*[6]

B. The anharmonic ratio of the three Symmedians and the Brocard diameter is numerically equal to C_1C_3/C_2C_3.

(10) The Lemoine axis 1 is perpendicular to s at the mid-point J of the segment II' (3), and since the segments MM' and II' are harmonic (5, cor.), the point J lies without the segment MM'[7]. Consequently:

The points of intersection of the Lemoine axis with the circumcircle are always imaginary.

COROLLARY. *The Lemoine point of a triangle lies always within the circumcircle* (9*A*).

(11) Since the line s passes through the pole O, with regard to γ_3, of the chord CC' (7), the pole P_3 of s with respect to γ_3 lies on CC'. On the other hand P_3 lies on 1, since s is perpendicular to the diameter 1 of γ_3. Similarly for the poles P_1, P_2 of s with respect to γ_1, γ_2. Hence:

The Symmedians meet the Lemoine axis in the respective poles of the Brocard diameter with regard to the Apollonian circles.

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[1] William Gallatly, *The Modern Geometry of the Triangle*, p. 6.

[2] Casey, *loc. cit.*, pp. 63, 64; Gallatly, *loc. cit.*, pp. 1, 2; R. Lachlan, *An. Elem. Treatise on Pure Geometry*, pp. 62, 63.

[3] John Casey, *Analytical Geometry*, p. 146, sec. ed. This MONTHLY, Feb., 1915, p. 59.

[4] Gallatly, *loc. cit.*, p. 5.

[5] Russell, *loc. cit.*, pp. 165, 117.

[6] William Gallatly, *The Modern Geometry of the Triangle*, p. 6.

[7] Russell, *loc. cit.*, p. 15.